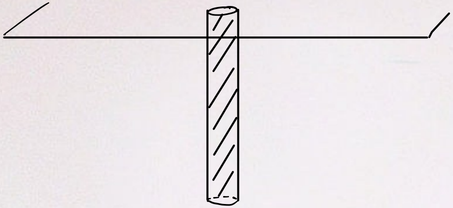
Lecture 1: Real World Problems and Differential Equations

Goals lecture of this lecture:

1. To get a brief idea of how real world problems are converted into equations;
2. To be convinced that real world problems can be formulated into equations consisting of derivatives (Differential equations)

Example 1: Elastic bar

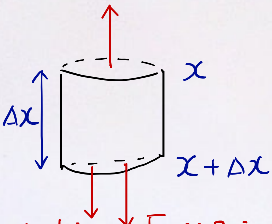


(Elastic bar hanged vertically under gravity)

Goal: Model the displacement $u(x)$ of the elastic bar at each position x under gravity.

Talk to "customers" (physicists):

Force : $c(x)\,\frac{du}{dx}(x)$



(A small portion of the elastic bar)

Force 1 : $\rho\,\Delta x\, a$

Force 2 : $c(x+\Delta x)\,\frac{du}{dx}(x+\Delta x)$

$$\text{Force 1} = \text{gravitational force}$$
$$= \underbrace{\rho}_{\text{density}}\,(\Delta x\,\underbrace{a}_{\text{cross-sectional area}})$$

At equilibrium state, all forces will be balanced.

$$\therefore \quad c(x+\Delta x)\,\frac{du}{dx}(x+\Delta x) \;-\; c(x)\,\frac{du}{dx}(x) \;+\; \big(\rho(x)\,\Delta x\, a\big)\, g \;=\; 0$$

Turn this formulation into an equation by dividing both sides by Δx and take $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} \frac{c(x+\Delta x)\,\frac{du}{dx}(x+\Delta x) - c(x)\,\frac{du}{dx}(x)}{\Delta x} + \rho(x)\, a\, g = 0$$

or

$$\left[\lim_{\Delta x \to 0} \frac{c(x)\,\frac{du}{dx}\Big|_{x+\Delta x} - c(x)\,\frac{du}{dx}\Big|_{x}}{\Delta x}\right] + \rho(x)\, a\, g = 0$$

$$\shortparallel$$

$$\frac{d}{dx}\left(c(x)\,\frac{du}{dx}\right)$$

$$\Leftrightarrow \quad -\frac{d}{dx}\left(c(x)\,\frac{du}{dx}\right) = \underbrace{\rho(x)\, a\, g}_{f(x)}$$

(Differential equation!)

Is a differential equation enough to determine the solution?

Consider a simple case. Let $c(x) \equiv 1$ and $f(x) = x^2 + 1$.

Then:

$$-\frac{d}{dx}\left(\overset{1}{c(x)}\,\frac{d\,u(x)}{dx}\right) = x^2 + 1$$

$$\int -\frac{d}{dx}\left(\frac{du(x)}{dx}\right)dx = \int (x^2 + 1)\,dx$$

$$\int -\frac{du}{dx} = \int \frac{x^3}{3} + x + C$$

$$-u(x) = \frac{x^4}{12} + \frac{x^2}{2} + Cx + D$$

Solution cannot be determined as it involves two unknown variables. Need more conditions!

What if we know $u(0) = u(1) = 0$ (fixing the two end points)

Then: $$u(x) = -\frac{x^4}{12} - \frac{x^2}{2} - Cx - D$$

$$\Rightarrow \quad u(0) = -D = 0 \quad \Rightarrow \quad D = 0$$

$$u(1) = -\frac{1}{12} - \frac{1}{2} - C = 0 \quad \Rightarrow \quad C = \frac{-7}{12}$$

$$\therefore \quad u(x) = -\frac{x^4}{12} - \frac{x^2}{2} + \frac{7}{12}x \quad \text{(unique sol)}$$

What if we know: $u(0) = 0$ and $\left.\dfrac{du}{dx}\right|_{x=1} = 0$.

Then:

$$u(x) = -\frac{x^4}{12} - \frac{x^2}{2} - Cx - D$$

$$\frac{du}{dx}(x) = -\frac{x^3}{3} - x - C$$

$$\Rightarrow \quad u(0) = -D = 0 \quad \Rightarrow \quad D = 0$$

$$\frac{du}{dx}(1) = -\frac{1}{3} - 1 - C = 0 \quad \Rightarrow \quad C = -\frac{4}{3}.$$

$$\therefore \quad u(x) = -\frac{x^4}{12} - \frac{x^2}{2} + \frac{4}{3}x \quad \text{(Unique solution)}$$

To determine a unique solution, we need more conditions!
(Need to ask "customers" what happens on the boundaries.)

A. Dirichlet : $u(0) = C_1$ and $u(1) = C_2$

(Does NOT involve derivatives)

B. Dirichlet + Neumann :

$u(0) = C_1$ and $c(x) \left.\dfrac{du}{dx}\right|_{x=1} = C_2$

(Neumann = involves derivatives)

Example 2: (Smooth approximation of unsmooth measurement)

Goal: Given a function (measurement) $w : [0,1] \to \mathbb{R}$, which is unsmooth. Find a smooth approximation $u : [0,1] \to \mathbb{R}$ of w, such that $u(0) = w(0) = 0$.

Rule: Unsmooth means $\left|\frac{du}{dx}\right|$ is big!

Mathematical formulation: (*)

Find $u : [0,1] \to \mathbb{R}$ with $u(0) = 0$ such that:

$$J(u) = \int_0^1 \left|\frac{du}{dx}\right| dx + \int_0^1 (u(x) - w(x))^2 \text{ is minimized}$$

This problem is related to solving:

$$-\frac{d}{dx}\left(\frac{\frac{du}{dx}}{\left|\frac{du}{dx}\right|}\right) + 2(u(x) - w(x)) = 0$$

(Differential eqt)

(Out of scope)

Analytic methods for solving differential equation

Note: Most differential equations do not have analytic (exact) solutions!

e.g.

$$-\frac{d}{dx}\Big(\underbrace{\overset{x}{c(x)}}_{\text{complicated}}\,u(x)\Big) = \underbrace{\overset{x^2}{f(x)}}_{\text{complicated}}$$

DOESN'T have analytic sol!

$\Big\downarrow$ convert / approximation

$$-\frac{d}{dx}\big(x\,u(x)\big) = x^2$$

(Have analytic solution! Give a good initial guess for the sol. of the original equation!)

Three most basic techniques:

(1) Integrating factor

(2) Separation of variables

(3) Analytic spectral (Fourier) method

(1) Integrating factor

(A) First order differential equation (involving first derivatives ONLY)

Consider: $(\ast)$ $\dfrac{dy}{dx} + P(x)\,y(x) = Q(x)$ (y is unknown function)

Let $M(x) = e^{\int_{s_0}^{x} P(s)\,ds}$ $\leftarrow$ s_0 some constant. Then, it is easy to check:

$$\frac{d}{dx}\big(M(x)\,y(x)\big) = \frac{dM(x)}{dx}\,y(x) + M(x)\frac{dy}{dx}$$

$$= \underbrace{e^{\int_{s_0}^{x} P(s)\,ds}}_{M(x)}\,P(x)\,y(x) + M(x)\frac{dy}{dx}$$

$$\therefore\ \frac{d}{dx}\big(M(x)\,y(x)\big) = M(x)\left(\frac{dy}{dx} + P(x)\,y(x)\right)$$

Multiply both sides of $(*)$ by $M(x)$:

$$\underbrace{M(x)\left(\frac{dy}{dx} + P(x)y(x)\right)}_{\frac{d}{dx}\left(M(x)y(x)\right)} = M(x)\,Q(x)$$

$$\therefore \quad \int \frac{d}{dx}\left(M(x)y(x)\right) = \int M(x)\,Q(x)$$

$$\Rightarrow \quad M(x)y(x) = \int M(x)\,Q(x)\,dx + C$$

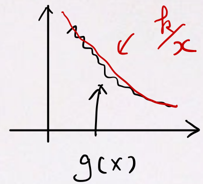
$$y(x) = \left[\int \left(e^{\int_{s_0}^{x} P(s)\,ds}\right) Q(x)\,dx + C\right]\left(e^{-\int_{s_0}^{x} P(s)\,ds}\right)$$

Remark: $M(x)$ is called the integrating factor.

Example 1: Consider : $\frac{dy}{dx} - g(x)\,y(x) = 0$, $1 \le x < \infty$ with $y(1) = 1$.

Suppose $g(x) \approx \frac{k}{x}$.

Find an approximated guess of $y(x)$.



Solution: Consider : $\frac{dy}{dx} - \frac{k}{x}\,y(x) = 0$

Let $M(x) = e^{\int -\frac{k}{x}\,dx} = e^{-k\ln x} = x^{-k}$

Then:
$$M(x)\left(\frac{dy}{dx} - \frac{k}{x}\,y(x)\right) = 0 \cdot M(x)$$

$$\Rightarrow \frac{d}{dx}\left(M(x)\,y(x)\right) = 0$$

$$\Rightarrow x^{-k} \approx M(x)\,y(x) = C \Rightarrow y(x) = Cx^{k}.$$

$\because$ $y(1) = 1 \Rightarrow 1 = C$. $\therefore$ $y(x) = x^k$ is an approximated guess of the solution.